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Generalized kappa-deformed spaces, star products and their realizations

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Abstract

In this work we investigate generalized kappa-deformed spaces. We develop a systematic method for constructing realizations of noncommutative (NC) coordinates as formal power series in the Weyl algebra. All realizations are related by a group of similarity transformations, and to each realization we associate a unique ordering prescription. Generalized derivatives, the Leibniz rule and coproduct, as well as the star product are found in all realizations. The star product and Drinfel'd twist operators are given in terms of the coproduct, and the twist operator is derived explicitly in special realizations. The theory is applied to a Nappi–Witten type of NC space.

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1. Introduction

Recently, there has been a growing interest in the formulation of physical theories on noncommutative (NC) spaces. The structures of such theories and their physical consequences were studied in [1–7]. Classification of NC spaces and investigation of their properties, in particular the development of a unifying approach to a generalized theory suitable for physical applications, is an important problem. In order to make a contribution in this direction we analyze a Lie algebra type NC space which is a generalized version of the kappa-deformed space.

Kappa-deformed spaces were studied by different groups, from both the mathematical and physical point of view [8–33]. There is also an interesting connection between the kappa-deformed spaces and doubly special relativity program [17, 18]. In a kappa-deformed space the noncommutative coordinates satisfy Lie algebra type relations depending on a deformation parameter $a \in \mathbb{R}^n$. The parameter a is on a very small length scale and yields the undeformed space when $\|a\| \to 0$. Other types of NC spaces frequently studied in the literature are the canonical theta-deformed spaces where the corresponding commutation relations are given by a second-rank antisymmetric tensor $\theta_{\mu\nu}$ (see [3, 4] and references therein).

A simple unification of kappa and theta-deformed spaces was first used in the study of the Wess–Zumino–Witten model [34]. Unification of these spaces was also the starting point in the algebraic study of the time-dependent NC geometry of a six-dimensional Cahen–Wallach pp-wave string background [35–37]. In this approach, the unification is achieved by adding a central element to the NC coordinates whose commutation relations are parameterized by real-valued parameters a and θ which are assumed to be equal.

The motivation for the present work is twofold. First, we want to generalize the unified kappa and theta-deformed spaces to arbitrary dimensions, and for arbitrary values of the parameters a and θ. Second, we want to develop a unifying approach to constructing realizations of such spaces in terms of ordinary commutative coordinates x_μ and derivatives ∂_μ which are convenient for physical applications. In the present work we assume that x_μ are coordinates in a Euclidean space, but the analysis can be easily extended to Minkowski space. We shall be mainly concerned with the Nappi–Witten type of NC space which arises in the study of pp-wave string background [37]. This space is made up of two copies of kappa-deformed space and one copy of theta-deformed space. Our analysis is based on the methods developed for algebras of deformed oscillators and the corresponding creation and annihilation operators [38–47]. The realization of a general Lie algebra type NC space in symmetric Weyl ordering was given in [48].

The outline of the paper and the summary of the main results is as follows. In section 2, we introduce a generalized kappa-deformed space of Nappi–Witten type, NW_4. We study realizations of the generators of NW_4 as formal power series with coefficients in the Weyl algebra. We show that there exist an infinite family of such realizations parameterized by two functions φ and F. For special choices of φ and F we obtain three important realizations: the right, symmetric left–right and Weyl realization. In section 3, we construct a group of similarity transformations acting transitively on the realizations. Section 4 deals with ordering prescriptions for NW_4. We show that to each realization one can associate an ordering prescription, and we find the prescriptions explicitly in terms of the parameter functions φ and F. In our approach the right, symmetric left–right and Weyl realizations correspond to the time, symmetric time and Weyl orderings defined in [37], respectively. Thus the orderings found in [37] are only special cases of an infinite family of ordering prescriptions for NW_4 constructed here. Furthermore, the time and symmetric time orderings can be viewed as limiting cases of an ordering prescription which interpolates between the two orderings.

In section 5, we consider the problem of extending the NC space NW_4 by generalized derivatives such that the extended space is a deformed Heisenberg algebra. We also define rotation operators on the extended space which generate the undeformed $so(4)$ algebra. The generalized derivatives and rotation operators are found in all realizations of NW_4. Section 6 deals with Leibniz rule and coproduct for the deformed Heisenberg algebra introduced in section 5. We find explicitly the coproduct depending on the parameter functions φ and F, and we give a relation between the coproducts in different realizations. Furthermore, star-product and Drinfel'd twist operators are considered in section 7. A general formula for the star product in terms of the coproduct is given, and an expression depending on φ and F is derived. Also, the corresponding twist operator is found explicitly for a wide class of realizations of NW_4. Finally, we describe how the obtained results generalize to higher dimensions.

2. Realizations of the Nappi–Witten space NW_4

Let us consider a unification of the canonical theta-deformed NC space and a Lie algebra type NC space with generators $X_1, X_2, \ldots, X_n$ and structure constants $C_{\mu\nu\lambda}$. Throughout the paper capital letters will be used consistently to denote NC coordinates. In order to include the theta-deformation given by a constant antisymmetric tensor $\theta_{\mu\nu}$, we introduce a central element X_0 such that

$$[X_\mu, X_\nu] = \mathrm{i}\theta_{\mu\nu}X_0 + \mathrm{i}\sum_\lambda C_{\mu\nu\lambda}X_\lambda. \tag{1}$$

The NC space defined by the commutation relations (1) is also of Lie algebra type provided all the Jacobi identities are satisfied,

$$\sum_\alpha (C_{\mu\nu\alpha}C_{\alpha\lambda\rho} + C_{\nu\lambda\alpha}C_{\alpha\mu\rho} + C_{\lambda\mu\alpha}C_{\alpha\nu\rho}) = 0, \tag{2}$$

$$\sum_\alpha (C_{\mu\nu\alpha}\theta_{\alpha\lambda} + C_{\nu\lambda\alpha}\theta_{\alpha\mu} + C_{\lambda\mu\alpha}\theta_{\alpha\nu}) = 0. \tag{3}$$

When $\theta_{\mu\nu} \to 0$ we obtain a Lie algebra type NC space with structure constants $C_{\mu\nu\lambda}$. Similarly, when $C_{\mu\nu\lambda} \to 0$ the space reduces to the canonical theta-deformed space with the additional central element X_0.

As an example consider a NC space with coordinates $X_+, X_-, Z_\mu, \bar{Z}_\mu$, $\mu = 1, \ldots, n$, satisfying the commutation relations:

$$[X_-, Z_\mu] = -\mathrm{i}aZ_\mu, \tag{4}$$

$$[X_-, \bar{Z}_\mu] = \mathrm{i}a\bar{Z}_\mu, \tag{5}$$

$$[Z_\mu, \bar{Z}_\nu] = \mathrm{i}2\theta\delta_{\mu\nu}X_+. \tag{6}$$

Here X_+ is the central element and $a, \theta \in \mathbb{R}$. We shall refer to the NC space defined by (4)–(6) as the generalized Nappi–Witten space NW_{2n+2}. In the special case when $a = \theta$ and $n = 2$, this space was recently studied by Halliday and Szabo in [37].

Without loss of generality we may assume that $n = 1$ since all the results are easily extended to $n > 1$. Thus, we shall consider the NC space NW_4 generated by X_+, X_-, Z and $\bar{Z}$ satisfying

$$[X_-, Z] = -\mathrm{i}aZ, \tag{7}$$

$$[X_-, \bar{Z}] = \mathrm{i}a\bar{Z}, \tag{8}$$

$$[Z, \bar{Z}] = \mathrm{i}2\theta X_+. \tag{9}$$

The space NW_4 may be considered a generalized kappa-deformed space since (9) defines a theta-deformation while (7) and (8) define two kappa-deformations. Since NW_4 is a Lie algebra, in future reference it will be denoted $\mathfrak{g}$.

For notational ease let $X = (X_+, X_-, Z, \bar{Z})$, and let $x = (x_+, x_-, z, \bar{z})$ be the ordinary commutative coordinates with the corresponding derivatives $\partial = (\partial_+, \partial_-, \partial_z, \partial_{\bar{z}})$. We seek a realization of the generators of $\mathfrak{g}$ as formal power series with coefficients in the Weyl algebra $\mathcal{A}_4$ generated by x and ∂. Let us consider realizations of the form

$$X_\mu = \sum_\alpha x_\alpha \phi_{\alpha\mu}(\partial), \qquad \phi_{\alpha\mu}(0) = \delta_{\alpha\mu}, \tag{10}$$

which are linear in x and $\phi_{\alpha\mu}(\partial)$ is a formal power series in ∂. We assume that there exists a dual relation

$$x_\mu = \sum_\alpha X_\alpha \Phi_{\alpha\mu}(\partial), \qquad \Phi_{\alpha\mu}(0) = \delta_{\alpha\mu}, \tag{11}$$

where $\Phi_{\alpha\mu}(\partial)$ is also a formal power series in ∂. A realization characterized by the functions $\phi_{\mu\nu}$ will be called a ϕ-realization. The generators of $\mathfrak{g}$ belong to $\overline{\mathcal{A}}_4$, the formal completion of $\mathcal{A}_4$. One may also consider realizations in which x_α is placed to the right of $\phi_{\alpha\mu}(\partial)$, or any linear combination of the two types. This is convenient when one requires Hermitian realizations. Indeed, let $\dagger\colon \mathcal{A}_4 \to \mathcal{A}_4$ be the Hermitian operator defined by $x_\mu^\dagger = x_\mu$, $\partial_\mu^\dagger = -\partial_\mu$ and $(x_\mu\partial_\nu)^\dagger = -\partial_\nu x_\mu$. If (10) is a realization, then

$$X_\mu = \frac{1}{2}\sum_\alpha (x_\alpha \phi_{\alpha\mu}(\partial) + \phi_{\alpha\mu}(-\partial)x_\alpha) \tag{12}$$

is a Hermitian realization since $X_\mu^\dagger = X_\mu$. Although such realizations are interesting in their own right, in this paper we shall restrict our attention to realizations of the type (10).

Let us assume the Ansatz

$$X_+ = x_+, \tag{13}$$

$$X_- = x_- + \mathrm{i}a[\bar{z}\partial_{\bar{z}}\gamma(A) - z\partial_z\gamma(-A)] + a\theta x_+\partial_z\partial_{\bar{z}}\psi(A), \tag{14}$$

$$Z = z\varphi(-A) + \mathrm{i}\theta x_+\partial_{\bar{z}}\eta(-A), \tag{15}$$

$$\bar{Z} = \bar{z}\varphi(A) - \mathrm{i}\theta x_+\partial_z\eta(A), \tag{16}$$

where $A = \mathrm{i}a\partial_-$, and the functions φ, η, γ and ψ satisfy the boundary conditions $\varphi(0) = 1$ and $\gamma(0), \psi(0)$ finite. The Ansatz (13)–(16) is of a fairly general nature leading to a number of interesting realizations discussed below. The boundary conditions ensure that in the limit $a, \theta \to 0$ the NC coordinates X_μ become the commutative coordinates x_μ.

Let us analyze the realization (13)–(16). Define $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ to be the antilinear map given by $\sigma(X_\pm) = X_\pm, \sigma(Z) = \bar{Z}$, and $\sigma(\bar{Z}) = Z$ which also preserves the Lie bracket, $\sigma([X_\mu, X_\nu]) = [\sigma(X_\mu), \sigma(X_\nu)]$. Then σ acts as a formal conjugation, and $\sigma^2 = \mathrm{i}d$. The action of σ on the generators of $\mathcal{A}_4$ is defined in the obvious way: $\sigma(x_\pm) = x_\pm, \sigma(z) = \bar{z}$, $\sigma(\bar{z}) = z$ and $\sigma(\partial_\pm) = \partial_\pm, \sigma(\partial_z) = \partial_{\bar{z}}, \sigma(\partial_{\bar{z}}) = \partial_z$, and $\sigma(x_\mu\partial_\nu) = \sigma(x_\mu)\sigma(\partial_\nu)$. The condition $\sigma(X_-) = X_-$ holds if and only if ψ is an even function, as seen from equation (14). The commutation relations (7)–(9) imply that the functions φ, η, γ and ψ are constrained by the system of equations

$$\gamma(A) - \frac{\varphi'(A)}{\varphi(A)} = 1, \tag{17}$$

$$\varphi(A)\eta(-A) + \varphi(-A)\eta(A) = 2, \tag{18}$$

$$\eta'(A) - \gamma(-A)\eta(A) - \psi(A)\varphi(A) + \eta(A) = 0, \tag{19}$$

where the prime denotes the differentiation with respect to A. It is convenient to introduce the auxiliary function

$$F(A) = \varphi(-A)\eta(A) - 1. \tag{20}$$

Then equation (18) implies that F is odd and, furthermore, $F = 0$ if and only if $\psi = 0$. For a given choice of φ and F one can uniquely determine the remaining functions η, γ and ψ.

Therefore, the Lie algebra $\mathfrak{g}$ admits infinitely many realizations parameterized by φ and F satisfying $\varphi(0) = 1$ and $F(0) = 0$.

Now we turn our attention to special realizations of $\mathfrak{g}$: the right realization, symmetric left–right and Weyl realization. As noted in the introduction to every realization one can associate an ordering prescription on the universal enveloping algebra $U(\mathfrak{g})$. The aforementioned realizations correspond to the time ordering, symmetric time ordering and Weyl symmetric ordering discussed in [37], respectively.

2.1. Special realizations

2.1.1. Special realization $\gamma = \gamma_0$. This subsection deals with the realization (13)–(16) when γ is a constant function, $\gamma = \gamma_0$, and $F = 0$. For this choice of the parameters equations (17)–(20) imply that $\varphi(A) = \eta(A) = \exp((\gamma_0 - 1)A)$ and $\psi(A) = 0$. Hence, the $\gamma = \gamma_0$ realization is given by

$$X_+ = x_+, \tag{21}$$

$$X_- = x_- + \mathrm{i}a\gamma_0(\bar{z}\partial_{\bar{z}} - z\partial_z), \tag{22}$$

$$Z = (z + \mathrm{i}\theta x_+\partial_{\bar{z}})\,\mathrm{e}^{-(\gamma_0-1)A}, \tag{23}$$

$$\bar{Z} = (\bar{z} - \mathrm{i}\theta x_+\partial_z)\,\mathrm{e}^{(\gamma_0-1)A}. \tag{24}$$

Of particular interest are the realizations with $\gamma_0 = 1$, $\gamma_0 = 1/2$ and $\gamma_0 = 0$:

(i) Right realization: $\gamma_0 = 1$

$$X_+ = x_+, \tag{25}$$

$$X_- = x_- + \mathrm{i}a(\bar{z}\partial_{\bar{z}} - z\partial_z), \tag{26}$$

$$Z = z + \mathrm{i}\theta x_+\partial_{\bar{z}}, \tag{27}$$

$$\bar{Z} = \bar{z} - \mathrm{i}\theta x_+\partial_z. \tag{28}$$

(ii) Symmetric left–right realization: $\gamma_0 = 1/2$

$$X_+ = x_+, \tag{29}$$

$$X_- = x_- + \frac{\mathrm{i}a}{2}(\bar{z}\partial_{\bar{z}} - z\partial_z), \tag{30}$$

$$Z = (z + \mathrm{i}\theta x_+\partial_{\bar{z}})\,\mathrm{e}^{\frac{1}{2}A}, \tag{31}$$

$$\bar{Z} = (\bar{z} - \mathrm{i}\theta x_+\partial_z)\,\mathrm{e}^{-\frac{1}{2}A}. \tag{32}$$

(iii) Left realization: $\gamma_0 = 0$

$$X_+ = x_+, \tag{33}$$

$$X_- = x_-, \tag{34}$$

$$Z = (z + \mathrm{i}\theta x_+\partial_{\bar{z}})\,\mathrm{e}^{A}, \tag{35}$$

$$\bar{Z} = (\bar{z} - \mathrm{i}\theta x_+\partial_z)\,\mathrm{e}^{-A}. \tag{36}$$

These realizations will be considered later in more detail when we establish a connection between realizations and ordering prescriptions.

2.1.2. Weyl realization. The Ansatz (13)–(16) also includes the so-called Weyl realization of $\mathfrak{g}$ which corresponds to the symmetric Weyl ordering on $U(\mathfrak{g})$. In this ordering all monomials in the basis of $U(\mathfrak{g})$ are completely symmetrized over all generators of $\mathfrak{g}$.

To this end we recall the following general result proved in [48]. Consider a Lie algebra over $\mathbb{C}$ with generators $X_1, X_2, \ldots, X_n$ and structure constants $C_{\mu\nu\lambda}$ satisfying

$$[X_\mu, X_\nu] = \mathrm{i}\sum_{\alpha=1}^{n} C_{\mu\nu\alpha} X_\alpha. \tag{37}$$

The Lie algebra (37) can be given a universal realization in terms of the commutative coordinates x_μ and derivatives ∂_μ, $1 \leqslant \mu \leqslant n$, as follows. Let $B = [B_{\mu\nu}]$ denote the $n \times n$ matrix of differential operators with elements

$$B_{\mu\nu} = \mathrm{i}\sum_{\alpha=1}^{n} C_{\alpha\nu\mu}\partial_\alpha, \tag{38}$$

and let $p(B) = B/(\exp(B) - 1)$ be the generating function for the Bernoulli numbers. Then, one can show that the generators of the Lie algebra (37) admit the realization

$$X_\mu = \sum_{\alpha=1}^{n} x_\alpha\, p(B)_{\alpha\mu}. \tag{39}$$

This is called the Weyl realization since it gives rise to the symmetric Weyl ordering on the enveloping algebra of (37).

We shall use the result (39) in order to obtain the Weyl realization of the Lie algebra $\mathfrak{g}$. Recall that the generators of $\mathfrak{g}$ are arranged as $X = (X_+, X_-, Z, \bar{Z})$; hence the structure constants $C_{\mu\nu\lambda}$ can be gleamed off from equations (7)–(9). Then equation (38) yields

$$B = \begin{pmatrix} 0 & 0 & -\mathrm{i}2\theta\partial_{\bar{z}} & \mathrm{i}2\theta\partial_z \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{i}a\partial_z & -\mathrm{i}a\partial_- & 0 \\ 0 & -\mathrm{i}a\partial_{\bar{z}} & 0 & \mathrm{i}a\partial_- \end{pmatrix}. \tag{40}$$

The explicit form of the matrix $p(B)$ can be found from the identity

$$p(B) = -\frac{B}{2} + \frac{B}{2}\coth\left(\frac{B}{2}\right). \tag{41}$$

One can show by induction that $B^{2n} = A^{2n-2}B^2$, $n \geqslant 1$, where $A = \mathrm{i}a\partial_-$, and

$$B^2 = \begin{pmatrix} 0 & 4a\theta\partial_z\partial_{\bar{z}} & \mathrm{i}2\theta\partial_{\bar{z}}A & \mathrm{i}2\theta\partial_z A \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i}a\partial_z A & A^2 & 0 \\ 0 & -\mathrm{i}a\partial_{\bar{z}} A & 0 & A^2 \end{pmatrix}. \tag{42}$$

Expanding equation (41) into Taylor series and collecting the terms with even powers of A we obtain

$$p(B) = 1 - \frac{B}{2} + h(A)B^2, \tag{43}$$

where we have defined

$$h(A) = \frac{1}{A^2}\left(\frac{A}{2}\coth\left(\frac{A}{2}\right) - 1\right). \tag{44}$$

Now equations (42) and (43) yield

$$p(B) = \begin{pmatrix} 1 & 4a\theta\partial_z\partial_{\bar z}h(A) & \mathrm{i}\theta\partial_{\bar z}(1+2Ah(A)) & -\mathrm{i}\theta\partial_z(1-2Ah(A)) \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{\mathrm{i}a}{2}\partial_z(1+2Ah(A)) & 1+\frac{A}{2}(1+2Ah(A)) & 0 \\ 0 & \frac{\mathrm{i}a}{2}\partial_{\bar z}(1-2Ah(A)) & 0 & 1-\frac{A}{2}(1-2Ah(A)) \end{pmatrix}. \tag{45}$$

Substituting equation (45) into equation (39) and simplifying, we obtain the Weyl realization

$$X_+ = x_+, \tag{46}$$

$$X_- = x_- + \mathrm{i}a\left[\bar z\partial_{\bar z}\left(\frac{1}{1-\mathrm{e}^{A}} + \frac{1}{A}\right) - z\partial_z\left(\frac{1}{1-\mathrm{e}^{-A}} - \frac{1}{A}\right)\right]$$
$$+ a\theta x_+\partial_z\partial_{\bar z}\frac{2}{A}\left(\coth\left(\frac{A}{2}\right) - \frac{2}{A}\right), \tag{47}$$

$$Z = z\frac{A}{1-\mathrm{e}^{-A}} + \mathrm{i}\theta x_+\partial_{\bar z}\left(\frac{2}{1-\mathrm{e}^{-A}} - \frac{2}{A}\right), \tag{48}$$

$$\bar Z = -\bar z\frac{A}{1-\mathrm{e}^{A}} - \mathrm{i}\theta x_+\partial_z\left(\frac{2}{1-\mathrm{e}^{A}} + \frac{2}{A}\right). \tag{49}$$

It is readily seen that the above realization is a special case of the original Ansatz (13)–(16) where

$$\varphi_s(A) = \frac{A}{\mathrm{e}^A - 1}, \tag{50}$$

$$\psi_s(A) = \frac{2}{A}\left(\coth\left(\frac{A}{2}\right) - \frac{2}{A}\right), \tag{51}$$

$$\gamma_s(A) = \frac{1}{A} - \frac{1}{\mathrm{e}^A - 1}, \tag{52}$$

$$\eta_s(A) = 2\gamma_s(A), \tag{53}$$

$$F_s(A) = \frac{A}{1-\cosh(A)} + \frac{\mathrm{e}^A + 1}{\mathrm{e}^A - 1}. \tag{54}$$

As required, these functions can be shown to satisfy the compatibility conditions (17)–(19).

3. Similarity transformations

In this section, we discuss similarity transformations which connect different realizations

$$X_\mu = \sum_{\alpha=1}^{n} x_\alpha\phi_{\alpha\mu}(\partial), \qquad \phi_{\alpha\mu}(0) = \delta_{\alpha\mu}. \tag{55}$$

The transformations act in a covariant way in the sense that the transformed realization is of the same type. These transformations can be used to generate new realizations of $\mathfrak{g}$ and new ordering prescriptions on $U(\mathfrak{g})$. They also relate the star products and coproducts in different realizations, as discussed in sections 6 and 7.

Let $\mathcal{A}_n$ denote the Weyl algebra generated by x_μ and ∂_μ, $1 \leqslant \mu \leqslant n$. Consider a differential operator S of the form

$$S = \exp\left(\sum_{\alpha=1}^{n} x_\alpha A_\alpha(\partial)\right), \tag{56}$$

where $A_\alpha(\partial)$ is a formal power series of $\partial = (\partial_1, \partial_2, \ldots, \partial_n)$. We assume that $A_\alpha(0) = 0$. Since the commutator of any two elements of the form $\sum_\alpha x_\alpha A_\alpha(\partial)$ is again of the same form, it follows from the Baker–Campbell–Hausdorff (BCH) formula that the family of operators S is a group under multiplication, with identity $S = 1$ when $A_\alpha = 0$ for all α. To each operator S we associate a similarity transformation $T_S: \overline{\mathcal{A}}_n \to \overline{\mathcal{A}}_n$, $T_S(u) = SuS^{-1}$. The transformations T_S form a subgroup of the group of inner automorphisms of $\overline{\mathcal{A}}_n$.

Let us examine the action of T_S on the generators of $\mathcal{A}_n$. If we denote $P = \sum_\alpha x_\alpha A_\alpha(\partial)$, then

$$T_S(x_\mu) = \exp(\mathrm{ad}(P))x_\mu. \tag{57}$$

By induction one can show that

$$\mathrm{ad}^k(P)x_\mu = \sum_{\alpha=1}^n x_\alpha R^{(k)}_{\alpha\mu}(\partial), \qquad k \geqslant 1, \tag{58}$$

where the functions $R^{(k)}_{\alpha\mu}$ are defined recursively by

$$R^{(1)}_{\alpha\mu}(\partial) = \frac{\partial A_\alpha}{\partial \partial_\mu}, \tag{59}$$

$$R^{(k)}_{\alpha\mu}(\partial) = \sum_{\beta=1}^n \left(\frac{\partial A_\alpha}{\partial \partial_\beta} R^{(k-1)}_{\beta\mu} + \frac{\partial R^{(k-1)}_{\alpha\mu}}{\partial \partial_\beta} A_\beta \right), \qquad k \geqslant 2. \tag{60}$$

Hence, we obtain

$$T_S(x_\mu) = \sum_{\alpha=1}^n x_\alpha \Psi_{\alpha\mu}(\partial), \tag{61}$$

where

$$\Psi_{\alpha\mu}(\partial) = \sum_{k=1}^\infty \frac{1}{k!} R^{(k)}_{\alpha\mu}(\partial) + \delta_{\alpha\mu}. \tag{62}$$

Similarly, the transformation of ∂_μ yields

$$T_S(\partial_\mu) = \partial_\mu + \sum_{k=1}^\infty \frac{(-1)^k}{k!} Q^{k-1}(A_\mu(\partial)), \tag{63}$$

where the operator Q is defined by

$$Q = \sum_{\alpha=1}^n A_\alpha(\partial) \frac{\partial}{\partial \partial_\alpha}. \tag{64}$$

We note that the transformation of ∂_μ is given only in terms of $\partial_1, \partial_2, \ldots, \partial_n$, which write symbolically as

$$T_S(\partial_\mu) = \Lambda_\mu(\partial). \tag{65}$$

The inverse transformations of x_μ and ∂_μ are of the same type,

$$T_S^{-1}(x_\mu) = \sum_{\alpha=1}^n x_\alpha \widetilde{\Psi}_{\alpha\mu}(\partial), \qquad T_S^{-1}(\partial_\mu) = \widetilde{\Lambda}_\mu(\partial). \tag{66}$$

The functions $\Psi_{\alpha\mu}$ and Λ_μ are related through the commutation relations for ∂_μ and x_ν. Substituting equations (61) and (65) into the commutator $[T_S(\partial_\mu), T_S(x_\nu)] = \delta_{\mu\nu}$ we find

$$\sum_{\alpha=1}^{n} \frac{\partial\Lambda_\nu}{\partial\partial_\alpha}\Psi_{\alpha\mu} = \delta_{\mu\nu}. \tag{67}$$

Define the vector $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$ and matrix $\Psi = [\Psi_{\mu\nu}]$. Then equation (67) implies that

$$\frac{\partial\Lambda}{\partial\partial} = \Psi^{-1}, \tag{68}$$

where

$$\frac{\partial\Lambda}{\partial\partial} = \left[\frac{\partial\Lambda_\nu}{\partial\partial_\mu}\right] \tag{69}$$

is the Jacobian of Λ.

To prove the covariance of the realization (55) under the action of T_S consider

$$T_S^{-1}(X_\mu) = \sum_{\alpha=1}^{n} T_S^{-1}(x_\alpha)\phi_{\alpha\mu}\big(T_S^{-1}(\partial)\big). \tag{70}$$

Using equation (66) the above expression becomes

$$T_S^{-1}(X_\mu) = \sum_{\beta=1}^{n} x_\beta\widetilde{\phi}_{\beta\mu}(\partial), \tag{71}$$

where

$$\widetilde{\phi}_{\beta\mu}(\partial) = \sum_{\alpha=1}^{n} \widetilde{\Psi}_{\beta\alpha}(\partial)\phi_{\alpha\mu}(\widetilde{\Lambda}(\partial)). \tag{72}$$

Let us introduce the new variables $y_\mu = T_S(x_\mu)$ and $\partial^y_\mu = T_S(\partial_\mu)$ (which also generate the Weyl algebra $\mathcal{A}_n$). Then equation (71) yields

$$X_\mu = \sum_{\beta=1}^{n} y_\beta\widetilde{\phi}_{\beta\mu}(\partial^y), \tag{73}$$

proving that the realization (55) is covariant under the change of variables $x_\mu \mapsto Sx_\mu S^{-1}$ and $\partial_\mu \mapsto S\partial_\mu S^{-1}$. Thus, the similarity transformation T_S maps the ϕ-realization (55) to $\widetilde{\phi}$-realization (73).

As an example, consider the right realization (25)–(28). It can be shown that the operator S mapping the right realization to the general Ansatz (13)–(16) (parameterized by φ and F) is given by

$$S = \exp\left(z\partial_z \ln[\varphi(-A)] + \bar{z}\partial_{\bar{z}} \ln\varphi(A) + \mathrm{i}\theta x_+\partial_z\partial_{\bar{z}}F(A)\frac{\ln\varphi(A)\varphi(-A)}{1-\varphi(A)\varphi(-A)}\right). \tag{74}$$

Direct calculation yields

$$\Lambda_+(\partial) = \partial_+ + \mathrm{i}\theta\partial_z\partial_{\bar{z}}\frac{F(A)}{\varphi(A)\varphi(-A)}, \tag{75}$$

$$\Lambda_-(\partial) = \partial_-, \tag{76}$$

$$\Lambda_z(\partial) = \partial_z\frac{1}{\varphi(-A)}, \tag{77}$$

$$\Lambda_{\bar{z}}(\partial) = \partial_{\bar{z}} \frac{1}{\varphi(A)}. \tag{78}$$

Now, the functions $\Psi_{\mu\nu}(\partial)$ can be calculated from equation (68). The group of transformations T_S acts transitively since any two realizations are related by $T_{S_1} T_{S_2}^{-1}$ where T_{S_i} maps the right realization to the (φ_i, F_i)-realization.

4. Generalized orderings

When considering the NC space NW_6 only three ordering prescriptions have been used in [37] for the construction of the corresponding star products: time ordering, symmetric time ordering and symmetric Weyl ordering. The time ordering is defined by

$$: \mathrm{e}^{\mathrm{i}kX} :_t = \mathrm{e}^{\mathrm{i}(k_z Z + k_{\bar{z}} \bar{Z})}\, \mathrm{e}^{\mathrm{i}k_- X_-}\, \mathrm{e}^{\mathrm{i}k_+ X_+}, \tag{79}$$

where we have denoted $k = (k_+, k_-, k_z, k_{\bar{z}}) \in \mathbb{R}^4$ and kX is the Euclidean space scalar product $kX = k_+X_+ + k_-X_- + k_z Z + k_{\bar{z}}\bar{Z}$. Since X_+ is the central element the position of $\mathrm{e}^{\mathrm{i}k_+X_+}$ is irrelevant. Here we consider only the Euclidean space, but the theory can be easily generalized to spaces with other signatures, e.g. the Minkowski space. The symmetric time and symmetric Weyl orderings are defined by

$$: \mathrm{e}^{\mathrm{i}kX} :_{st} = \mathrm{e}^{\mathrm{i}\frac{1}{2}k_- X_-}\, \mathrm{e}^{\mathrm{i}(k_z Z + k_{\bar{z}} \bar{Z})}\, \mathrm{e}^{\mathrm{i}\frac{1}{2}k_- X_-}\, \mathrm{e}^{\mathrm{i}k_+ X_+} \tag{80}$$

and

$$: \mathrm{e}^{\mathrm{i}kX} :_s = \mathrm{e}^{\mathrm{i}(k_+X_+ + k_-X_- + k_z Z + k_{\bar{z}}\bar{Z})}, \tag{81}$$

respectively. We note that the orderings are determined by the position of X_- in the monomial basis of $U(\mathfrak{g})$. For illustration, the monomials of order three (modulo X_+) in the time ordering are

$$\begin{aligned} &X_-^3,\, Z^3,\, \bar{Z}^3,\, Z^2X_-,\, \bar{Z}^2X_-,\, ZX_-^2,\, \bar{Z}X_-^2,\, \tfrac{1}{2}(Z\bar{Z} + \bar{Z}Z)X_-, \\ &\tfrac{1}{3}(Z^2\bar{Z} + Z\bar{Z}Z + \bar{Z}Z^2),\, \tfrac{1}{3}(Z\bar{Z}^2 + \bar{Z}Z\bar{Z} + \bar{Z}^2 Z). \end{aligned} \tag{82}$$

The corresponding monomials in the symmetric time ordering and symmetric Weyl ordering are given by

$$\begin{aligned} &X_-^3,\, Z^3,\, \bar{Z}^3,\, \tfrac{1}{2}(Z^2X_- + X_-Z^2),\, \tfrac{1}{2}(\bar{Z}^2X_- + X_-\bar{Z}^2),\, \tfrac{1}{2}(ZX_-^2 + X_-^2 Z), \\ &\tfrac{1}{2}(\bar{Z}X_-^2 + X_-\bar{Z}),\, \tfrac{1}{3}(Z^2\bar{Z} + Z\bar{Z}Z + \bar{Z}Z^2),\, \tfrac{1}{3}(Z\bar{Z}^2 + \bar{Z}Z\bar{Z} + \bar{Z}^2 Z), \\ &\tfrac{1}{4}(Z\bar{Z}X_- + X_-Z\bar{Z} + \bar{Z}ZX_- + X_-\bar{Z}Z), \end{aligned} \tag{83}$$

and

$$\begin{aligned} &X_-^3,\, Z^3,\, \bar{Z}^3,\, \tfrac{1}{3}(X_-^2 Z + X_-ZX_- + ZX_-^2),\, \tfrac{1}{3}(X_-^2\bar{Z} + X_-\bar{Z}X_- + \bar{Z}X_-^2), \\ &\tfrac{1}{3}(Z^2\bar{Z} + Z\bar{Z}Z + \bar{Z}Z^2),\, \tfrac{1}{3}(\bar{Z}^2 Z + \bar{Z}Z\bar{Z} + Z\bar{Z}^2),\, \tfrac{1}{3}(X_-Z^2 + ZX_-Z + Z^2X_-), \\ &\tfrac{1}{3}(X_-\bar{Z}^2 + \bar{Z}X_-\bar{Z} + \bar{Z}X_-),\, \tfrac{1}{6}(X_-Z\bar{Z} + \text{cyclic perm.}), \end{aligned} \tag{84}$$

respectively. In future reference the time ordering and symmetric time ordering will be called the right ordering and symmetric left–right ordering, respectively, as implied by the position of X_- in the monomial basis.

In this section, we show that to each realization (13)–(16) of the generators of $\mathfrak{g}$ one can associate an ordering prescription on $U(\mathfrak{g})$. This leads to an infinite family of ordering prescriptions parameterized by the functions φ and F. In our approach the orderings used in [37] appear as special cases corresponding to the right, symmetric left–right and Weyl realization found in section 2.

Let us begin by defining the 'vacuum' state

$$|0\rangle = 1, \qquad \partial_\mu|0\rangle = 0. \tag{85}$$

Let X_μ^ϕ denote the generator X_μ in ϕ-realization (55), and let X_μ^s denote the Weyl realization of X_μ. It has been shown in [25, 27] that for kappa-deformed spaces a simple relation holds

$$\mathrm{e}^{\mathrm{i}kX^s}|0\rangle = \mathrm{e}^{\mathrm{i}kx}, \qquad k \in \mathbb{R}^n. \tag{86}$$

Equation (86) can be generalized to any ϕ-realization by requiring

$$\mathrm{e}^{\mathrm{i}K(k)X^\phi}|0\rangle = \mathrm{e}^{\mathrm{i}kx} \tag{87}$$

for some function $K:\mathbb{R}^n \to \mathbb{R}^n$. Let T_S be the similarity transformation mapping the ϕ-realization (55) to the symmetric Weyl realization

$$X_\mu^s = \sum_\alpha y_\alpha \phi_{\alpha\mu}^s(\partial^y), \tag{88}$$

where the variables y_μ and ∂_μ^y are given by equations (61) and (65). In this realization we have

$$\mathrm{e}^{\mathrm{i}K(k)X^s}|0\rangle = \mathrm{e}^{\mathrm{i}K(k)y}, \tag{89}$$

hence $\mathrm{e}^{\mathrm{i}K(k)y} = \mathrm{e}^{\mathrm{i}kx}$. Since $\partial_\mu^y = \Lambda_\mu(\partial)$, it follows that $\partial_\mu^y\,\mathrm{e}^{\mathrm{i}K(k)y} = \Lambda_\mu(\partial)\,\mathrm{e}^{\mathrm{i}kx}$ which implies

$$K_\mu(k) = -\mathrm{i}\Lambda_\mu(\mathrm{i}k), \qquad 1 \leqslant \mu \leqslant n. \tag{90}$$

Thus, K_μ is completely determined by the similarity transformation T_S mapping the ϕ-realization to the Weyl realization.

For each ϕ-realization we define the ϕ-ordering by

$$:\mathrm{e}^{\mathrm{i}kX}:_\phi = \mathrm{e}^{\mathrm{i}K(k)X}, \tag{91}$$

where K is given by equation (90). If X is represented in ϕ-realization, then

$$:\mathrm{e}^{\mathrm{i}kX}:_\phi |0\rangle = \mathrm{e}^{\mathrm{i}kx}. \tag{92}$$

The above expression gives a simple relation between a ϕ-realization and ϕ-ordering. The monomial basis for $U(\mathfrak{g})$ in ϕ-ordering can be explicitly derived from equation (91). Let $m = (m_i)$ be a multi-index with $m_i \in \mathbb{N}_0$, and let

$$\left(\frac{\partial}{\partial k}\right)^m = \frac{\partial^{|m|}}{\partial k_1^{m_1}\cdots\partial k_n^{m_n}}. \tag{93}$$

A basis element of order $|m|$ is given by

$$P_m(X) = \left(-\mathrm{i}\frac{\partial}{\partial k}\right)^m \mathrm{e}^{\mathrm{i}K(k)X}\Big|_{k=0}. \tag{94}$$

Since $K(0) = 0$, $P_m(X)$ is a polynomial of degree $|m|$. In the Weyl realization when $K(k) = k$, equation (94) leads to the Weyl ordering whereby the polynomials $P_m(X)$ are completely symmetrized over the generators of $\mathfrak{g}$.

Let us illustrate the above ideas by computing an ordering prescription for the NC space NW_4. For a general realization parameterized by φ and F we have

$$:\mathrm{e}^{\mathrm{i}kX}:_{(\varphi,F)} = \mathrm{e}^{\mathrm{i}(K_+(k)X_+ + K_-(k)X_- + K_z(k)Z + K_{\bar{z}}(k)\bar{Z})}. \tag{95}$$

One can use equations (75)–(78) to find the similarity transformation mapping the (φ, F)-realization to Weyl realization. Then, equation (90) yields

$$K_+(k) = k_+ - \theta k_z k_{\bar{z}} \frac{F(-ak_-) - F_s(-ak_-)}{\varphi(-ak_-)\varphi(ak_-)}, \tag{96}$$

$$K_-(k) = k_-, \tag{97}$$

$$K_z(k) = k_z \frac{\varphi_s(ak_-)}{\varphi(ak_-)}, \tag{98}$$

$$K_{\bar{z}}(k) = k_{\bar{z}} \frac{\varphi_s(-ak_-)}{\varphi(-ak_-)}, \tag{99}$$

where φ_s and F_s are the parameter functions defined by equations (50) and (54). Thus, equations (95)–(99) define an infinite family of orderings on $U(\mathfrak{g})$ depending on the parameter functions ϕ and F.

Of particular interest is the realization $\gamma = \gamma_0$, in which case

$$\varphi(ak_-) = \exp((\gamma_0 - 1)ak_-), \qquad F(ak_-) = 0. \tag{100}$$

In this realization the function K becomes

$$K(k) = \left(k_+ + \theta k_z k_{\bar{z}} F_s(-ak_-), k_-, k_z \frac{\varphi_s(ak_-)}{\varphi(ak_-)}, k_{\bar{z}} \frac{\varphi_s(-ak_-)}{\varphi(-ak_-)}\right). \tag{101}$$

It can be shown that the ordering induced by this realization can be written in exponential form as

$$\mathrm{e}^{\mathrm{i}(1-\gamma_0)k_- X_-}\, \mathrm{e}^{\mathrm{i}(k_z Z + k_{\bar{z}} \bar{Z})}\, \mathrm{e}^{\mathrm{i}\gamma_0 k_- X_-} = \exp\left[\mathrm{i}\left(k_- X_- + k_z \frac{\varphi_s(ak_-)}{\varphi(ak_-)} Z + k_{\bar{z}} \frac{\varphi_s(-ak_-)}{\varphi(-ak_-)} \bar{Z}\right)\right], \tag{102}$$

where the central element X_+ has been left out. The above ordering has three interesting cases: the right ordering for $\gamma_0 = 1$, left ordering for $\gamma_0 = 0$ and symmetric left–right ordering for $\gamma_0 = 1/2$. Therefore, equation (102) may be interpreted as an interpolation between the left and right ordering. A comparison with equations (79)–(80) shows that the right and symmetric left–right orderings are precisely the time and symmetric time orderings constructed in [37].

5. Generalized derivatives

This section is devoted to extensions of the Lie algebra $\mathfrak{g}$ defined by (7)–(9) by addition of generalized derivatives. The motivation for considering such extensions is to extend the deformation of the commutative space to the entire phase space. For a general Lie algebra type NC space a detailed treatment of the generalized derivatives may be found in [49]. Here we consider extensions of $\mathfrak{g}$ such that the generalized derivatives and the Lie algebra $\mathfrak{g}$ are complementary subalgebras of a deformed Heisenberg algebra $\mathfrak{h}$. A natural way to define $\mathfrak{h}$ is as follows. If the generators of $\mathfrak{g}$ are given by ϕ-realization (10), we define the generalized derivative D_μ by setting $D_\mu = \partial_\mu$. Then the commutation relations yield $[D_\mu, D_\nu] = 0$ and $[D_\mu, X_\nu] = \phi_{\mu\nu}(D)$. Furthermore, the Jacobi identities are satisfied for all combinations of the generators X_μ and D_ν. In the limit as $a, \theta \to 0$ we have $[D_\mu, X_\nu] = \delta_{\mu\nu}$, hence $\mathfrak{h}$ is a deformed Heisenberg algebra. Obviously, there are infinitely many such extensions depending on the realization ϕ. In our case the simplest extension is obtained in the right realization (25)–(28) when $\phi_{\mu\nu}$ are linear in D_μ. Then $\mathfrak{h}$ is a Lie algebra defined by the commutation relations (7)–(9) and

$$[D_+, X_+] = 1, \qquad [D_+, Z] = \mathrm{i}\theta D_{\bar{z}}, \tag{103}$$

$$[D_+, X_-] = 0, \qquad [D_+, \bar{Z}] = -\mathrm{i}\theta D_z, \tag{104}$$

$$[D_-, X_+] = 0, \qquad [D_-, Z] = 0, \tag{105}$$

$$[D_-, X_-] = 1, \qquad [D_-, \bar{Z}] = 0, \tag{106}$$

$$[D_z, X_+] = 0, \qquad [D_z, Z] = 1, \tag{107}$$

$$[D_z, X_-] = -\mathrm{i}aD_z, \qquad [D_z, \bar{Z}] = 0, \tag{108}$$

$$[D_{\bar{z}}, X_+] = 0, \qquad [D_{\bar{z}}, Z] = 0, \tag{109}$$

$$[D_{\bar{z}}, X_-] = \mathrm{i}aD_{\bar{z}}, \qquad [D_{\bar{z}}, \bar{Z}] = 1. \tag{110}$$

This algebra agrees with the deformed Heisenberg algebra discussed in [37]. We note that if any other realization is used for the construction of $\mathfrak{h}$, then $\mathfrak{h}$ will not be of Lie type since $\phi_{\mu\nu}(D)$ is generally a formal power series in D_μ.

Now let us fix the algebra $\mathfrak{h}$ as above. We seek realizations of $\mathfrak{h}$ when the NC coordinates X_μ are given by the general Ansatz (13)–(16). Accordingly, we no longer have $D_\mu = \partial_\mu$ since the realizations of D_μ must be modified in order to satisfy the commutators (103)–(110). First, we consider the derivatives D_z and $D_{\bar{z}}$. Assume that $D_z = \partial_z G(A)$ where $G(A)$ is a formal power series in A satisfying the boundary condition $G(0) = 1$. The boundary condition ensures that $D_z \to \partial_z$ as $a \to 0$. Using equations (14) and (17) one can show that $[D_z, X_-] = -\mathrm{i}aD_z$ if and only if

$$\frac{G'(A)}{G(A)} = \frac{\varphi'(-A)}{\varphi(-A)}. \tag{111}$$

The unique solution of this equation satisfying $G(0) = 1$ is given by $G(A) = 1/\varphi(-A)$. Hence,

$$D_z = \partial_z \frac{1}{\varphi(-A)}. \tag{112}$$

One can verify that if D_z is given by equation (112), then the remaining commutators with D_z are automatically satisfied. A similar argument yields

$$D_{\bar{z}} = \partial_{\bar{z}} \frac{1}{\varphi(A)}. \tag{113}$$

Next we consider D_+. The relation $[D_+, X_+] = 1$ implies that

$$D_+ = \partial_+ + H(\partial_-, \partial_z, \partial_{\bar{z}}) \tag{114}$$

for some function H. Substituting equations (14) and (114) into the commutator $[D_+, X_-] = 0$ we obtain

$$a\theta\partial_z\partial_{\bar{z}}\psi(A) + \frac{\partial H}{\partial\partial_-} + \mathrm{i}a\frac{\partial H}{\partial\partial_{\bar{z}}}\partial_{\bar{z}}\gamma(A) - \mathrm{i}a\frac{\partial H}{\partial\partial_z}\partial_z\gamma(-A) = 0. \tag{115}$$

Similarly, the commutator $[D_+, Z] = \mathrm{i}\theta D_{\bar{z}}$ yields

$$\mathrm{i}\theta\partial_{\bar{z}}\eta(-A) + \frac{\partial H}{\partial\partial_z}\varphi(-A) = \mathrm{i}\theta\partial_{\bar{z}}\frac{1}{\varphi(A)}. \tag{116}$$

The structure of equation (115) suggests that H is of the form $H = \partial_z\partial_{\bar{z}}H_0(A)$. Inserting this expression into equation (116) we obtain

$$\mathrm{i}\theta\eta(-A)\varphi(A) + H_0(A)\varphi(A)\varphi(-A) = \mathrm{i}\theta. \tag{117}$$

Equations (117) and (20) imply that $H_0(A)\varphi(A)\varphi(-A) = \mathrm{i}\theta F(A)$, which yields

$$H = \mathrm{i}\theta\partial_z\partial_{\bar{z}}\frac{F(A)}{\varphi(A)\varphi(-A)}. \tag{118}$$

One can verify that the above expression for H is consistent with equation (115). Therefore, D_+ is given by

$$D_+ = \partial_+ + \mathrm{i}\theta\partial_z\partial_{\bar{z}}\frac{F(A)}{\varphi(A)\varphi(-A)}. \tag{119}$$

As required, the remaining commutator $[D_+, \bar{Z}] = -\mathrm{i}\theta D_z$ is automatically satisfied. Finally, we observe that the relations (105) and (106) trivially hold if we define $D_- = \partial_-$, hence D_- is the same in all realizations.

For future reference we collect the realizations of D_μ obtained here:

$$D_+ = \partial_+ + \mathrm{i}\theta\partial_z\partial_{\bar{z}}\frac{F(A)}{\varphi(A)\varphi(-A)}, \tag{120}$$

$$D_- = \partial_-, \tag{121}$$

$$D_z = \partial_z\frac{1}{\varphi(-A)}, \tag{122}$$

$$D_{\bar{z}} = \partial_{\bar{z}}\frac{1}{\varphi(A)}. \tag{123}$$

Comparing the expressions for D_μ with equations (75)–(78) we conclude that the generalized derivatives are given by the similarity transformation $D_\mu = S\partial_\mu S^{-1}$ where S is defined by equation (74). We note that for $\varphi(A) = 1$ and $F(A) = 0$ (right realization) we obtain $D_\mu = \partial_\mu$, as required.

The Lie algebra $\mathfrak{h}$ can be extended further by adding the rotation generators $M_{\mu\nu}$ which form the ordinary rotation algebra $so(4)$. The rotation generators are defined by $M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$, and satisfy

$$M_{\mu\nu} = -M_{\nu\mu}, \tag{124}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \delta_{\nu\lambda}M_{\mu\rho} - \delta_{\mu\lambda}M_{\nu\rho} - \delta_{\nu\rho}M_{\mu\lambda} + \delta_{\mu\rho}M_{\nu\lambda}. \tag{125}$$

Suppose that X_μ is represented in the right realization (25)–(28), and let $D_\mu = \partial_\mu$. By solving equations (25)–(28) for x_μ the rotation generators can be expressed in terms of X_μ and D_ν as

$$M_{+-} = X_+D_- - X_-D_+ - \mathrm{i}a(ZD_z - \bar{Z}D_{\bar{z}})D_+ - 2a\theta X_+D_+D_zD_{\bar{z}}, \tag{126}$$

$$M_{+z} = X_+D_z - ZD_+ + \mathrm{i}\theta X_+D_+D_{\bar{z}}, \tag{127}$$

$$M_{+\bar{z}} = X_+D_{\bar{z}} - \bar{Z}D_+ - \mathrm{i}\theta X_+D_+D_z, \tag{128}$$

$$M_{-z} = X_-D_z - ZD_- + \mathrm{i}a(ZD_z - \bar{Z}D_{\bar{z}})D_z + \mathrm{i}\theta X_+D_-D_{\bar{z}} + 2a\theta X_+D_z^2D_{\bar{z}}, \tag{129}$$

$$M_{-\bar{z}} = X_-D_{\bar{z}} - \bar{Z}D_- - \mathrm{i}a(\bar{Z}D_{\bar{z}} - ZD_z)D_{\bar{z}} - \mathrm{i}\theta X_+D_-D_z + 2a\theta X_+D_zD_{\bar{z}}^2, \tag{130}$$

$$M_{z\bar{z}} = ZD_{\bar{z}} - \bar{Z}D_z - \mathrm{i}\theta X_+\left(D_z^2 + D_{\bar{z}}^2\right). \tag{131}$$

Now the commutators $[M_{\mu\nu}, X_\lambda]$ and $[M_{\mu\nu}, D_\lambda]$ can be easily found, which we omit here. In this case the commutators $[M_{\mu\nu}, X_\lambda]$ are not linear in $M_{\mu\nu}$ and X_λ. We note, however, that by choosing a different realization of X_μ the rotation generators may be constructed so that the commutators $[M_{\mu\nu}, X_\lambda]$ are of Lie algebra type. For a different approach to the construction of $M_{\mu\nu}$ in kappa-deformed spaces see [9, 10, 25, 27].

6. The Leibniz rule and coproduct

Having introduced generalized derivatives we now set to find the Leibniz rule for D_μ and the corresponding coproduct ΔD_μ. The generalized derivative D_μ induces a linear map $D_\mu : U(\mathfrak{g}) \to U(\mathfrak{g})$ defined as follows. For $1, X_\mu \in U(\mathfrak{g})$ we define $D_\mu \bullet 1 = 0$ and $D_\mu \bullet X_\nu = [D_\mu, X_\nu] \bullet 1$, where $[D_\mu, X_\nu] \bullet 1$ is calculated using the commutation relations (103)–(110). The action of D_μ on monomials of higher degree can be defined inductively. Suppose we have defined $D_\mu \bullet f(X)$ where $f(X) \in U(\mathfrak{g})$ is a monomial of order n. Then the action of D_μ on a monomial of order $n+1$ is defined by

$$D_\mu \bullet (X_\nu f(X)) = X_\nu D_\mu \bullet f(X) + [D_\mu, X_\nu] \bullet f(X). \tag{132}$$

Since $[D_\mu, X_\nu]$ is given by equations (103)–(110), by induction hypothesis $[D_\mu, X_\nu] \bullet f(X)$ is well defined.

One can show that the Leibniz rule for D_μ is given by

$$\begin{aligned} D_+ \bullet (f(X)g(X)) = {} & (D_+ \bullet f(X))g(X) + f(X)\,(D_+ \bullet g(X)) \\ & + \mathrm{i}\theta\,\mathrm{e}^{A}\,(D_z \bullet f(X))\,(D_{\bar z} \bullet g(X)) \\ & - \mathrm{i}\theta\,\mathrm{e}^{-A}\,(D_{\bar z} \bullet f(X))\,(D_z \bullet g(X))\,, \end{aligned} \tag{133}$$

$$D_- \bullet (f(X)g(X)) = (D_- \bullet f(X))\,g(X) + f(X)\,(D_- \bullet g(X))\,, \tag{134}$$

$$D_z \bullet (f(X)g(X)) = (D_z \bullet f(X))\,g(X) + \mathrm{e}^{-A} f(X)\,(D_z \bullet g(X))\,, \tag{135}$$

$$D_{\bar z} \bullet (f(X)g(X)) = (D_{\bar z} \bullet f(X))\,g(X) + \mathrm{e}^{A} f(X)\,(D_{\bar z} \bullet g(X))\,, \tag{136}$$

where $A = \mathrm{i}aD_-$. From the above relations one obtains the corresponding coproduct

$$\Delta D_+ = D_+ \otimes 1 + 1 \otimes D_+ + \mathrm{i}\theta\,\mathrm{e}^{A} D_z \otimes D_{\bar z} - \mathrm{i}\theta\,\mathrm{e}^{-A} D_{\bar z} \otimes D_z, \tag{137}$$

$$\Delta D_- = D_- \otimes 1 + 1 \otimes D_-, \tag{138}$$

$$\Delta D_z = D_z \otimes 1 + \mathrm{e}^{-A} \otimes D_z, \tag{139}$$

$$\Delta D_{\bar z} = D_{\bar z} \otimes 1 + \mathrm{e}^{A} \otimes D_{\bar z}. \tag{140}$$

If $a = \theta$, this coproduct agrees with the time-ordered coproduct found in [37].

Now let us consider the algebra generated by X_μ and ∂_μ where X_μ is given by the Ansatz (13)–(16). The generators satisfy the commutation rules:

$$[\partial_+, X_+] = 1, \qquad [\partial_+, Z] = \mathrm{i}\theta\partial_{\bar z}\eta(-A), \tag{141}$$

$$[\partial_+, X_-] = a\theta\partial_z\partial_{\bar z}\psi(A), \qquad [\partial_+, \bar Z] = -\mathrm{i}\theta\partial_z\eta(A), \tag{142}$$

$$[\partial_-, X_+] = 0, \qquad [\partial_-, Z] = 0, \tag{143}$$

$$[\partial_-, X_-] = 1, \qquad [\partial_-, \bar Z] = 0, \tag{144}$$

$$[\partial_z, X_+] = 0, \qquad [\partial_z, Z] = \varphi(-A), \tag{145}$$

$$[\partial_z, X_-] = -\mathrm{i}a\partial_z\gamma(-A), \qquad [\partial_z, \bar Z] = 0, \tag{146}$$

$$[\partial_{\bar z}, X_+] = 0, \qquad [\partial_{\bar z}, Z] = 0, \tag{147}$$

$$[\partial_{\bar z}, X_-] = \mathrm{i}a\partial_{\bar z}\gamma(A), \qquad [\partial_{\bar z}, \bar Z] = \varphi(A). \tag{148}$$

This is also a deformed Heisenberg algebra, as seen by taking the limit $a, \theta \to 0$. The coproduct $\Delta\partial_\mu$ can be found from the coproduct of D_μ and equations (120)–(123) relating D_μ and ∂_μ. From equation (121) we have

$$\Delta\partial_- = \partial_- \otimes 1 + 1 \otimes \partial_-, \tag{149}$$

which implies that $\Delta A = A \otimes 1 + 1 \otimes A$. For convenience let us denote $A_1 = A \otimes 1$ and $A_2 = 1 \otimes A$, so that $\Delta A = A_1 + A_2$. Furthermore, using equations (122) and (139) we find

$$\begin{aligned}\Delta\partial_z &= \Delta D_z \Delta\varphi(-A) \\ &= \varphi(-A_1 - A_2)\left(\frac{\partial_z}{\varphi(-A)} \otimes 1 + \mathrm{e}^{-A} \otimes \frac{\partial_z}{\varphi(-A)}\right),\end{aligned} \tag{150}$$

where we have used commutativity of D_z and $\varphi(A)$. Similarly, one obtains

$$\Delta\partial_{\bar z} = \varphi(A_1 + A_2)\left(\frac{\partial_{\bar z}}{\varphi(A)} \otimes 1 + \mathrm{e}^{A} \otimes \frac{\partial_{\bar z}}{\varphi(A)}\right). \tag{151}$$

It follows from equation (120) that

$$\Delta\partial_+ = \Delta D_+ - \mathrm{i}\theta\Delta\partial_z\Delta\partial_{\bar z}\frac{F(A_1 + A_2)}{\varphi(A_1 + A_2)\varphi(-A_1 - A_2)}. \tag{152}$$

Inserting equations (150) and (151) into the above expression we obtain

$$\begin{aligned}\Delta\partial_+ = \Delta D_+ - \mathrm{i}\theta&\left(\frac{\partial_z}{\varphi(-A)} \otimes 1 + \mathrm{e}^{-A} \otimes \frac{\partial_z}{\varphi(-A)}\right) \\ &\times \left(\frac{\partial_{\bar z}}{\varphi(A)} \otimes 1 + \mathrm{e}^{A} \otimes \frac{\partial_{\bar z}}{\varphi(A)}\right) F(A_1 + A_2).\end{aligned} \tag{153}$$

If the coproduct ΔD_+ in equation (137) is expressed in terms of ∂_μ using equations (120)–(123), then after simplifying one can show that $\Delta\partial_+$ takes the form

$$\begin{aligned}\Delta\partial_+ = \partial_+ \otimes 1 + 1 \otimes \partial_+ &+ \mathrm{i}\theta\frac{\partial_z\partial_{\bar z}}{\varphi(A)\varphi(-A)} \otimes 1(F(A) \otimes 1 - F(A_1 + A_2)) \\ &+ \mathrm{i}\theta 1 \otimes \frac{\partial_z\partial_{\bar z}}{\varphi(A)\varphi(-A)}(1 \otimes F(A) - F(A_1 + A_2)) \\ &+ \mathrm{i}\theta\, \mathrm{e}^{A}\frac{\partial_z}{\varphi(-A)} \otimes \frac{\partial_{\bar z}}{\varphi(A)}(1 \otimes 1 - F(A_1 + A_2)) \\ &- \mathrm{i}\theta\, \mathrm{e}^{-A}\frac{\partial_{\bar z}}{\varphi(A)} \otimes \frac{\partial_z}{\varphi(-A)}(1 \otimes 1 + F(A_1 + A_2)).\end{aligned} \tag{154}$$

The coproduct $\Delta\partial_\mu$ is fixed by the realization (φ, F). If φ and F parameterize the Weyl realization (cf equations (50)–(54)) and $a = \theta$, then $\Delta\partial_\mu$ yields the Weyl ordered coproduct found in [37]. Recall that all realizations are related by similarity transformations described in section 3. Thus, if the coproduct is known in one realization, then it is known in any other realization. Hence, $\Delta\partial_\mu$ is unique in the sense that there is only one equivalence class $[\Delta\partial_\mu]$ containing all the coproducts found above.

7. Star products and twists

In this section, we study isomorphisms between the spaces of smooth functions of commutative coordinates x_μ and NC coordinates X_μ. These isomorphisms are defined in terms of ϕ-realizations of X_μ given by equation (10).

We define the ϕ-induced isomorphism Ω_ϕ by

$$\Omega_\phi f(x) = f(x)|0\rangle \equiv \hat{f}_\phi(X). \tag{155}$$

Here $f(x)|0\rangle$ is calculated by expressing x_μ in terms of X_μ and ∂_μ from equation (11), and placing ∂_μ to the far right using the commutation relations $[\partial_\mu, X_\nu] = \phi_{\mu\nu}(\partial)$. Similarly, the inverse map is defined by

$$\Omega_\phi^{-1}\hat{f}(X) = \hat{f}(X)|0\rangle \equiv f_\phi(x), \tag{156}$$

where $\hat{f}(X)|0\rangle$ is calculated using equation (10).

We define the ϕ-star product of functions $f(x)$ and $g(x)$ by

$$(f \star_\phi g)(x) = \hat{f}_\phi(X)\hat{g}_\phi(X)|0\rangle. \tag{157}$$

The star product can be written in terms of the isomorphism Ω_ϕ as

$$(f \star_\phi g)(x) = \big(\Omega_\phi f(x)\big)\, g(x), \tag{158}$$

where the derivatives ∂_μ in $\Omega_\phi(f(x))$ are placed to the far right and act on the function $g(x)$. The star product may also be written in the form

$$(f \star_\phi g) = m_0\mathcal{F}_\phi(f \otimes g), \tag{159}$$

where m_0 denotes the ordinary pointwise multiplication and $\mathcal{F}_\phi$ is the corresponding Drinfel'd twist operator [37]. We introduce the ϕ-deformed multiplication $m_\phi = m_0\mathcal{F}$ so that $f \star_\phi g = m_\phi(f \otimes g)$. The isomorphism Ω_ϕ can be written in terms of the twist operator $\mathcal{F}_\phi$ as

$$(\Omega_\phi f)g = m_0\mathcal{F}_\phi(f \otimes g) \quad \forall g. \tag{160}$$

Let us now consider the following problem. Given the exponential functions $\mathrm{e}^{\mathrm{i}kx}$ and $\mathrm{e}^{\mathrm{i}qx}$, $k, q \in \mathbb{R}^n$, we want to calculate their star product in ϕ-realization. For NC coordinates X_μ we have

$$\mathrm{e}^{\mathrm{i}kX}\,\mathrm{e}^{\mathrm{i}qX} = \mathrm{e}^{\mathrm{i}D_s(k,q)X}, \tag{161}$$

where the function $D_s\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ can be found in principle from the Dynkin form of the BCH formula. If X is represented in the symmetric Weyl realization, denoted X^s, then $\mathrm{e}^{\mathrm{i}kX^s}|0\rangle = \mathrm{e}^{\mathrm{i}kx}$. This implies that

$$\mathrm{e}^{\mathrm{i}kx} \star_s \mathrm{e}^{\mathrm{i}qx} = \mathrm{e}^{\mathrm{i}D_s(k,q)x}, \tag{162}$$

where $\star_s$ denotes the Weyl-ordered star product. The above relations can be generalized to arbitrary ϕ-ordering,

$$:\mathrm{e}^{\mathrm{i}kX}:_\phi :\mathrm{e}^{\mathrm{i}qX}:_\phi =: \mathrm{e}^{\mathrm{i}D_\phi(k,q)X}:_\phi, \tag{163}$$

$$\mathrm{e}^{\mathrm{i}kx} \star_\phi \mathrm{e}^{\mathrm{i}qx} = \mathrm{e}^{\mathrm{i}D_\phi(k,q)x}, \tag{164}$$

for some function $D_\phi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Let us find the correspondence between D_ϕ and D_s. Recall from equation (91) that in ϕ-ordering we have

$$:\mathrm{e}^{\mathrm{i}kX}:_\phi = \mathrm{e}^{\mathrm{i}K_\phi(k)X}, \tag{165}$$

where K_ϕ is given by the similarity transformation Λ which maps the ϕ-ordering to symmetric Weyl ordering. Furthermore, it follows from equation (161) that

$$:\mathrm{e}^{\mathrm{i}kX}:_\phi :\mathrm{e}^{\mathrm{i}qX}:_\phi = \mathrm{e}^{\mathrm{i}D_s(K_\phi(k),K_\phi(q))X}. \tag{166}$$

In view of equation (165) we have

$$\mathrm{e}^{\mathrm{i}D_s(K_\phi(k),K_\phi(q))X} =: \mathrm{e}^{\mathrm{i}K_\phi^{-1}\left(D_s(K_\phi(k),K_\phi(q))\right)X}:_\phi, \tag{167}$$

hence

$$:\mathrm{e}^{\mathrm{i}kX}:_\phi :\mathrm{e}^{\mathrm{i}qX}:_\phi =: \mathrm{e}^{\mathrm{i}K_\phi^{-1}\left(D_s(K_\phi(k),K_\phi(q))\right)X}:_\phi\,. \tag{168}$$

Therefore, the function D_ϕ is given by

$$D_\phi(k,q) = K_\phi^{-1}(D_s(K_\phi(k), K_\phi(q))). \tag{169}$$

If the isomorphism Ω_ϕ is restricted to the space of Schwartz functions on $\mathbb{R}^n$, then for a Schwartz function $f(x)$ we may define the Fourier transform

$$\widetilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{170}$$

In this case, $\hat{f}_\phi(X) = \Omega_\phi f(x)$ has the Fourier representation

$$\hat{f}_\phi(X) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widetilde{f}(k) : \mathrm{e}^{\mathrm{i}kX} :_\phi \mathrm{d}k \tag{171}$$

defined in terms of the ϕ-ordering $: \mathrm{e}^{\mathrm{i}kX} :_\phi$. Using the Fourier representation (171) and equation (164) it can be shown that the general ϕ-ordered star product can be expressed in terms of a bi-differential operator as

$$(f \star_\phi g)(x) = \mathrm{e}^{\mathrm{i}x[D_\phi(-\mathrm{i}\partial^u, -\mathrm{i}\partial^v) + \mathrm{i}\partial_u + \mathrm{i}\partial^v]} f(u)g(v)\big|_{\substack{u=x\\ v=x}}. \tag{172}$$

Next we want to relate the coproduct Δ_ϕ in ϕ-realization with the function $D_\phi(k,q)$. Let us start with the undeformed coproduct Δ_0 satisfying

$$\partial_\mu m_0 = m_0 \Delta_0 \partial_\mu, \tag{173}$$

which gives a simple relation between the Leibniz rule for ∂_μ and the coproduct Δ_0. The above equation implies that

$$\partial_\mu m_\phi = m_\phi \Delta_\phi \partial_\mu, \tag{174}$$

where $\Delta_\phi = \mathcal{F}_\phi^{-1}\Delta_0\mathcal{F}_\phi$ and $\mathcal{F}_\phi$ is the twist operator in ϕ-realization. It follows from equations (164) and (174) that

$$m_\phi \Delta_\phi \partial_\mu (\mathrm{e}^{\mathrm{i}kx} \otimes \mathrm{e}^{\mathrm{i}qx}) = \mathrm{i}D_\phi(k,q)_\mu\, \mathrm{e}^{\mathrm{i}kx} \star_\phi \mathrm{e}^{\mathrm{i}qx}, \tag{175}$$

where $D_\phi(k,q)_\mu$ denotes the μ-component of $D_\phi(k,q)$. The coproduct $\Delta_\phi\partial_\mu$ has a generic form

$$\Delta_\phi \partial_\mu = \sum_\alpha A^{(1)}_{\alpha\mu}(\partial) \otimes A^{(2)}_{\alpha\mu}(\partial), \tag{176}$$

thus using the above expression we find

$$m_\phi \Delta_\phi \partial_\mu (\mathrm{e}^{\mathrm{i}kx} \otimes \mathrm{e}^{\mathrm{i}qx}) = \left(\sum_\alpha A^{(1)}_{\alpha\mu}(\mathrm{i}k) A^{(2)}_{\alpha\mu}(\mathrm{i}q) \right) \mathrm{e}^{\mathrm{i}kx} \star_\phi \mathrm{e}^{\mathrm{i}qx}. \tag{177}$$

Therefore, comparing equations (175) and (177) we conclude

$$D_\phi(k,q)_\mu = -\mathrm{i}\widetilde{\Delta}_\phi(\mathrm{i}k, \mathrm{i}q)_\mu, \tag{178}$$

where we have denoted

$$\widetilde{\Delta}_\phi(\mathrm{i}k, \mathrm{i}q)_\mu = \sum_\alpha A^{(1)}_{\alpha\mu}(\mathrm{i}k) A^{(2)}_{\alpha\mu}(\mathrm{i}q). \tag{179}$$

Thus, equation (178) gives a correspondence between the coproduct Δ_ϕ in ϕ-realization and the function D_ϕ. Using this relation the star product in ϕ-realization may be written as

$$(f \star_\phi g)(x) = m_0\left\{\mathrm{e}^{x(\Delta_\phi - \Delta_0)\partial^u} f(u) \otimes g(u)\right\}_{u=x}. \tag{180}$$

In the right, symmetric left–right and Weyl realizations equation (180) agrees with the corresponding star products in [37].

Let us find a relation between the star products in different realizations. Let T_S be the similarity transformation which maps the ϕ_1-realization to ϕ_2-realization. Recall that S is explicitly given by $S = S_2 S_1^{-1}$ where S_i is of the form (74), and T_{S_i} maps the right realization to ϕ_i-realization. Fix $\hat{f}(X) \in U(\mathfrak{g})$, and let $X_\mu^{(i)}$ denote X_μ in ϕ_i-realization. Then $\hat{f}(X^{(i)})|0\rangle = f_i(x)$. Since $X_\mu^{(2)} = S X_\mu^{(1)} S^{-1}$ we have

$$\hat{f}\big(X_\mu^{(2)}\big) = S\hat{f}(X^{(1)})S^{-1}|0\rangle = S\hat{f}(X^{(1)}), \tag{181}$$

where we have used $S^{-1}|0\rangle = 1$. Therefore, $f_2(x) = Sf_1(x)$ which implies that the star products in two realizations are related by

$$f \star_{\phi_2} g = S(S^{-1} f \star_{\phi_1} S^{-1} g). \tag{182}$$

Using equation (178) one can deduce the function $D_\phi = \big(D_\phi^{(+)}, D_\phi^{(-)}, D_\phi^{(z)}, D_\phi^{(\bar{z})}\big)$ for the NC space NW_4 from the coproduct $\Delta_\phi \partial_\mu$ found in section 6,

$$\begin{aligned} D_\phi^{(+)}(k,q) = k_+ + q_+ &+ \theta \frac{k_z k_{\bar{z}}}{\varphi(ak_-)\varphi(-ak_-)}[F(ak_-) - F(ak_- + aq_-)] \\ &+ \theta \frac{q_z q_{\bar{z}}}{\varphi(aq_-)\varphi(-aq_-)}[F(aq_-) - F(ak_- + aq_-)] \\ &- \theta\, \mathrm{e}^{-ak_-} \frac{k_z q_{\bar{z}}}{\varphi(ak_-)\varphi(-aq_-)}[1 + F(ak_- + aq_-)] \\ &+ \theta\, \mathrm{e}^{ak_-} \frac{k_{\bar{z}} q_z}{\varphi(-ak_-)\varphi(aq_-)}[1 - F(ak_- + aq_-)], \end{aligned} \tag{183}$$

$$D_\phi^{(-)}(k,q) = k_- + q_-, \tag{184}$$

$$D_\phi^{(z)}(k,q) = \varphi(ak_- + aq_-)\left[\frac{k_z}{\varphi(ak_-)} + \mathrm{e}^{ak_-}\frac{q_z}{\varphi(aq_-)}\right], \tag{185}$$

$$D_\phi^{(\bar{z})}(k,q) = \varphi(-ak_- - aq_-)\left[\frac{k_{\bar{z}}}{\varphi(-ak_-)} + \mathrm{e}^{-ak_-}\frac{q_{\bar{z}}}{\varphi(-aq_-)}\right]. \tag{186}$$

Deformed addition in ϕ-ordering of the momenta k and q is defined by

$$k \oplus_\phi q = D_\phi(k,q). \tag{187}$$

The binary operation $\oplus_\phi$ depends on the ϕ-realization and represents a deformation of ordinary addition since

$$k \oplus_\phi q = k + q + O(a,\theta). \tag{188}$$

This non-Abelian operation is associative, which follows from equation (164) and associativity of the star product. The neutral element is $0 \in \mathbb{R}^n$ since

$$k \oplus_\phi 0 = D_\phi(k,0) = K_\phi^{-1}(D_s(K_\phi(k),0)) = K_\phi^{-1}(K_\phi(k)) = k, \tag{189}$$

and similarly $0 \oplus_\phi k = k$. The inverse element, denoted by $\underline{k}$, satisfies

$$k \oplus_\phi \underline{k} = \underline{k} \oplus_\phi k = 0. \tag{190}$$

It follows from equation (188) that

$$\underline{k} = -k + O(a,\theta), \tag{191}$$

hence $\underline{k}$ is a deformation of the ordinary opposite element $-k$, and it is the antipode of k. The inverse element $\underline{k}$ can be found from the condition $D_\phi(k, \underline{k}) = 0$,

$$\underline{k} = \left(-k_+, -k_-, -k_z\, \mathrm{e}^{-ak_-}\frac{\varphi(-ak_-)}{\varphi(ak_-)}, -k_{\bar{z}}\mathrm{e}^{ak_-}\frac{\varphi(ak_-)}{\varphi(-ak_-)}\right). \tag{192}$$

In view of equation (164) we have

$$\mathrm{e}^{\mathrm{i}kx} \star_\phi \mathrm{e}^{\mathrm{i}\underline{k}x} = 1. \tag{193}$$

Note that in the Weyl realization when φ is given by equation (50) we have $\underline{k} = -k$. In fact, the same is true for any function of the form $\varphi(k) = \mathrm{e}^{-\frac{k}{2}}\widetilde{\varphi}(k)$ where $\widetilde{\varphi}$ is an even function.

We conclude this section by giving the explicit form of the twist operator $\mathcal{F}_\phi$ in two special cases: (i) $\theta = 0, a \neq 0$ and (ii) $a = 0, \theta \neq 0$. When $\theta = 0$ and $a \neq 0$ the twist operator is given by

$$\begin{aligned}\mathcal{F}_\phi = \exp\Bigg[&(\bar{z}\partial_{\bar{z}} \otimes 1)\ln\frac{\varphi(A_1 + A_2)}{\varphi(A_1)} + (z\partial_z \otimes 1)\ln\frac{\varphi(-A_1 - A_2)}{\varphi(-A_1)}\\ &+ (1 \otimes \bar{z}\partial_{\bar{z}})\left(A_1 + \ln\frac{\varphi(A_1 + A_2)}{\varphi(A_2)}\right) + (1 \otimes z\partial_z)\left(-A_1 + \ln\frac{\varphi(-A_1 - A_2)}{\varphi(-A_2)}\right)\Bigg],\end{aligned} \tag{194}$$

where we have denoted $A_1 = A \otimes 1$ and $A_2 = 1 \otimes A$, as before. Recall that $\varphi(A) = \exp((\gamma_0 - 1)A)$. Hence, in the right realization ($\gamma_0 = 1$) the operator $\mathcal{F}$ has a simple form

$$\mathcal{F}_R = \exp[A \otimes (\bar{z}\partial_{\bar{z}} - z\partial_z)]. \tag{195}$$

The twist operators for $\gamma_0 = 0$ and $\gamma_0 = 1$ were previously constructed in [25], and are given by equations (59) and (60), respectively. For $\gamma_0 = 1/2$ the twist operator agrees with the twist proposed by Bu *et al* [26].

On the other hand, when $a = 0$ and $\theta \neq 0$ we find

$$\mathcal{F} = \exp\left[\frac{\theta}{2}(x_+\partial_z \otimes \partial_{\bar{z}} + \partial_z \otimes x_+\partial_{\bar{z}} - x_+\partial_{\bar{z}} \otimes \partial_z - \partial_{\bar{z}} \otimes x_+\partial_z)\right]. \tag{196}$$

In all cases the twist operator satisfies the cocycle condition [50, 51]

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}. \tag{197}$$

For kappa-deformed spaces in n dimensions this relation was proved in [52]. We remark that the kappa-deformation of Poincaré symmetries cannot be described by Drinfeld twist as an element of the tensor product of two enveloping Poincaré algebras. The twist operator considered here is embedded in a larger algebra $U(\mathrm{igl}(4)) \otimes U(\mathrm{igl}(4))$ where igl(4) is the inhomogeneous general linear algebra.

8. Concluding remarks

We have investigated a generalized kappa-deformed space of Nappi–Witten type which is a unification of kappa and theta-deformed spaces with arbitrary deformation parameters a and θ. We have constructed an infinite family of realizations of this space in terms of commutative coordinates x_μ in Euclidean space and the corresponding derivatives ∂_μ. All realizations are related by a group of similarity transformations defined by the operator (74). In particular, we have investigated a class of realizations (13)–(16) parameterized by functions φ and F. To each realization we have associated a corresponding ordering prescription given by equation (95). For a special choice of φ and F, and with $a = \theta$, we reproduce the time ordering, symmetric time ordering and Weyl ordering constructed by Halliday and Szabo [37]. Unlike [37], in our approach the ordering prescriptions follow from a general procedure and they are all related by similarity transformations.

Furthermore, we have extended the space of NC coordinates by introducing generalized derivatives. We have shown that to each realization of the NC coordinates one can associate an extended phase space which is a deformed Heisenberg algebra. The simplest extension is obtained in the right realization when the extended algebra $\mathfrak{h}$ is of Lie type, and it agrees with the deformed Heisenberg algebra discussed in [37]. The algebra $\mathfrak{h}$ was further extended by introducing rotation operators $M_{\mu\nu}$ which satisfy the ordinary $so(4)$ algebra. The coproduct $\Delta M_{\mu\nu}$ is not closed in the tensor product of the enveloping algebras of $so(4)$. Hence, if one wishes to have the coalgebra structure then it is natural to consider a larger algebra igl(4) [26].

The Leibniz rule and coproduct was found for the extended algebra $\mathfrak{h}$. Applying the similarity transformations to this coproduct we derived the Leibniz rule and coproduct for all realizations described by φ and F. In the right realization, symmetric left–right and Weyl realization this coproduct agrees with the coproducts considered in [37] when $a = \theta$. We derived a general formula for the star product in terms of the coproduct, and we found an explicit expression for the star product in all realizations parameterized by φ and F. In the above-mentioned special realizations these results reproduce the star products found in [37]. Also, we have found a general form of the Drinfel'd twist operator for special values of the deformation parameters: (i) $a \neq 0, \theta = 0$ and (ii) $a = 0, \theta \neq 0$. The twist operator is embedded in the tensor product of two enveloping algebras of igl(4).

The results in this paper are easily generalized to higher dimensions when the NC space under consideration is generated by X_+, X_- and n copies of Z and $\bar{Z}$ (see equations (4)–(6)). In this case the Weyl algebra $\mathcal{A}_4$ is replaced by $\mathcal{A}_{2n+2}$ and all expressions involving realizations, ordering prescriptions, Leibniz rules, coproducts and twist operators should be modified as follows. If z, $\bar{z}$, ∂_z and $\partial_{\bar{z}}$ appear linearly, they are replaced by z_μ, $\bar{z}_\mu$, ∂_{z_μ} and $\partial_{\bar{z}_\mu}$, respectively. Any quadratic combinations involving $z\partial_z$, $\bar{z}\partial_{\bar{z}}$ and $\partial_z\partial_{\bar{z}}$ are replaced by sums over μ. Finally, we remark that all the results obtained here can be easily extended to Mikonwski space.

Our general formalism can be further developed in order to construct and analyze QFT on NC spaces generalizing the results obtained by Halliday and Szabo [37] by applying twist operators in a systematic way. These problems will be addressed in future work.

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